Distance dependence of radiation energy flux

1992 J. Phys. A: Math. Gen. 25 6371

(http://iopscience.iop.org/0305-4470/25/23/030)

J. Phys. A: Math. Gen. 25 (1992) 6371-6377. Printed in the UK

Distance dependence of radiation energy flux

N P Klepikov and I M Ternov

Faculty of Physics, Moscow State University, 117234 Moscow, Russia

Received 14 February 1992, in final form 23 June 1992

Abstract. The distance dependence of the energy flux and of the wave polarization in incoherent synchrotron radiation, as well as the form of the separatrix dividing the near and wave zones, are elaborated.

The classical theory of the electromagnetic radiation of charged particles has a century-long history. But until the past few decades little attention seems to have been paid to the energy flux in regions of space not belonging to the wave zone, and to the structure of the closed surface (separatrix) separating the wave zone from the near zone (for the case of relativistic particles). Teitelboim *et al* (1980) investigated carefully the distance dependence of the field components, the components of the energy-momentum tensor and the components of the angular momentum tensor for an accelerated particle. Subsequently Villaroel and Fuenzalida (1987) and Villaroel and Millan (1988) considered some special effects that may be observed near the orbit of the particle. Unfortunately, their interpretation of the formulae obtained, as we shall show below, is incorrect, and this compelled us to consider the problem again and in more detail.

There is a number of reasons why the results of Villaroel *et al* are misleading.

First, they misinterpreted the results of their predecessors. In particular, neither Schott (1912) nor Schwinger (1949) mentioned the radiation in the 'radial' direction; instead, they considered the radiation in *all* directions starting from the point where the particle is at the moment of radiation, and integrated the radiation intensity over the whole sphere. Also, the allegation that the energy of the particle does not play any role in Schott's formula is wrong since this formula leads to the correct energy dependence of the total radiation intensity.

Second, Villaroel *et al* did not make a clear distinction between the notions of the radiation and the energy flux. The determination of the radiation intensity is not at all based on some approximations, but is *by definition* that part of the energy flux which within a fixed solid angle penetrates *any* distant region of space. Liénard (1898) was the first to use this definition. Such a definition is reasonable since otherwise we might say that a charge moving uniformly and rectilinearly also radiates. Indeed, in the latter case the energy density rises in those regions of space which the charge approaches, while it falls in those from which it moves off, but these energy fluxes do not reach distant regions of space. Therefore, there is a local energy flux, but no radiation.

Third, the only natural physical parameter responsible for the behaviour of the energy flux is the distance $\mathcal{R}$ between the radiation and the observation points; the dimensionless parameter is $a/\mathscr{R}$, where a is the radius of curvature of the particle trajectory at the radiation point. Instead, Villaroel *et al* use the parameter $\xi = a/r$, where r is the distance of the observation point from the centre of the orbit of the charge. Obviously, $a/\mathscr{R} = \xi/(1-\xi^2)^{1/2}$, and these parameters are, in principle, equivalent. Nevertheless, we shall see that all expansions in $a/\mathscr{R}$ are finite, while those in ξ are infinite and quite complicated. This makes difficult the correct interpretation of the results obtained. In their formula (2.7) Villaroel and Fuenzalida neglect the term S_2, the only term introducing the distance dependence of the energy flux. Therefore, there should remain no distance dependence. Finally, Villaroel *et al* did not indicate the principles of the organization of an experiment intended to observe the radiation in the 'radial' direction. Obviously, the expected results essentially depend on whether or not the detector is protected against the radiation in the tangential direction.

Everything concerning the energy flux refers also to the fluxes of linear and angular momenta.

If the charge e_1 is situated at the 4-point x_1, it is known to create at any field point $x = (\boldsymbol{r}, t)$ the Liénard–Wiechert field

$$F_{ij}^{(1)}(x) = \frac{e_1}{[(x-x_1)u_1]^3}\{[(x-x_1)_i u_{1j} - (x-x_1)_j u_{1i}][1+(x-x_1)w_1] - [(x-x_1)_i w_{1j} - (x-x_1)_j w_{1i}][(x-x_1)u_1]\} \tag{1}$$

$$(x-x_1)^2 = 0 \tag{2}$$

where u_1 and w_1 are the 4-velocity and the 4-acceleration of the charge at x_1 (we are using a complex Euclidian metric). Equation (1) is an exact solution of Maxwell's equations, but is expressed implicitly, since it is applicable only when (2) is satisfied.

The field on the wavefront may be expressed explicitly if we fix the radiation point x_1 and do the substitution

$$x_i = x_{1i} + \mathscr{R}\mathfrak{n}_i \tag{3}$$

where $\mathfrak{n}_i = (\boldsymbol{n}, \mathrm{i}n_0)$ is a 4-vector satisfying the condition $\mathfrak{n}^2 = 0$. Under a Lorentz transformation $\mathfrak{n}$ is transformed as the linear momentum of a photon. Equation (2) is now satisfied identically. The field acquires the form (in (4)–(7) we omit the subscript 1)

$$F_{ij} = \frac{e}{\mathscr{R}(\mathfrak{n}u)^3}(\mathfrak{n}_i f_j - \mathfrak{n}_j f_i) + \frac{e}{\mathscr{R}^2(\mathfrak{n}u)^3}(\mathfrak{n}_i u_j - \mathfrak{n}_j u_i) \tag{4}$$

where

$$f_i = u_i(\mathfrak{n}w) - w_i(\mathfrak{n}u) \qquad \mathfrak{n}f = 0. \tag{5}$$

Note that there is no infinite expansion in the powers of $1/\mathscr{R}$. An infinite expansion arises if the radiating system consists of more than one particle (Klepikov 1985a, b). But we are not going to consider this case here.

In order to prevent unjustified objections we should note that the fact that (4) is a solution of Maxwell's equations cannot be tested by means of differentiating (4) by the independent variables ($\mathscr{R}$ and two angles of $\boldsymbol{n}$), since any variation of the observation point entails, according to (2), a variation of the radiation point.

Now we may consider the expressions for the electric and magnetic fields ($\beta = v/c$):

$$\boldsymbol{E} = \frac{e}{\mathcal{R}}\left(\frac{(\boldsymbol{n}-\boldsymbol{v}/c)\boldsymbol{n}\boldsymbol{w} - \boldsymbol{w}(1-\boldsymbol{n}\boldsymbol{v}/c)}{c^2(1-\boldsymbol{n}\boldsymbol{v}/c)^3} + \frac{1}{\mathcal{R}}\frac{(\boldsymbol{n}-\boldsymbol{v}/c)(1-\beta^2)}{(1-\boldsymbol{n}\boldsymbol{v}/c)^3}\right) \tag{6}$$

$$\boldsymbol{H} = -\frac{e}{\mathcal{R}}\left(\frac{(1/c)(\boldsymbol{n}\times\boldsymbol{v})(\boldsymbol{n}\boldsymbol{w}) + (\boldsymbol{n}\times\boldsymbol{w})(1-\boldsymbol{n}\boldsymbol{v}/c)}{c^2(1-\boldsymbol{n}\boldsymbol{v}/c)^3} + \frac{(\boldsymbol{n}\times\boldsymbol{v})(1-\beta^2)}{c\mathcal{R}(1-\boldsymbol{n}\boldsymbol{v}/c)^3}\right). \tag{7}$$

The total energy flux at the distance $\mathcal{R}$ from the radiation point is expressed as

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t_1} = \frac{c}{4\pi}\mathcal{R}^2\int \mathrm{d}\Omega\, \boldsymbol{n}(\boldsymbol{E}\times\boldsymbol{H})\left(1-\frac{\boldsymbol{n}\boldsymbol{v}}{c}\right) \tag{8}$$

where the last factor is the derivative $\mathrm{d}t/\mathrm{d}t_1$.

Angles of the direction $\boldsymbol{n}$ may be introduced in a number of ways. In any case the result of the integration is

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t_1} = \frac{2ce^2}{3a^2}\left(\frac{\beta^4}{(1-\beta^2)^2} + \frac{a^2\beta^2}{\mathcal{R}^2(1-\beta^2)}\right) \tag{9}$$

where a is the radius of curvature of the trajectory of the charge at the radiation point. For synchrotron radiation the values of a and β^2 may be considered as independent of t_1.

The first term of (9) is called the radiation intensity and was first found by Liénard. The second one is the local energy flux, which raises the energy density in some parts of the local region surrounding the position of the charge.

If, instead of integrating (8) over the whole sphere, we consider only the plane tangential to the trajectory of the charge at x_1, and integrate (8) over the azimuth in this plane, we get

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t_1} = \frac{ce^2}{a^2}\left(\frac{\beta^4(4+3\beta^2)}{16\mathcal{R}(1-\beta^2)^{5/2}} + \frac{a^2\beta^2(4+\beta^2)}{16\mathcal{R}^3(1-\beta^2)^{3/2}}\right). \tag{10}$$

The reader should note that in (10) the energy flux is integrated not over a solid angle, but over a plane angle in the tangent plane. Since $a/\mathcal{R} = \xi/(1-\xi^2)^{1/2}$, for $1-\beta^2 \ll 1$ and ξ near to unity the first term of (10) coincides with formula (2.46) of Villaroel and Fuenzalida. Consequently, their formula describes the radiation intensity not in a solid angle but in a plane angle also. However, any detector occupies some solid angle. To account for this, (10) should be multiplied by $\mathcal{R}$, and any increase in the radiation intensity near the orbit (which Villaroel *et al* insist on) disappears.

From (9) we may deduce the criterion of the wave zone in a radiation process. The second term of (9) is much less than the first if

$$\mathcal{R} \gg a\frac{(1-\beta^2)^{1/2}}{\beta} = \frac{\lambda_0}{2\pi}(1-\beta^2)^{1/2} \tag{11}$$

where λ_0 is the wavelength corresponding to the rotation frequency of the charge. For sufficiently high energy of the particles, the radius of the wave zone may be much less than this wavelength.

Take into account that (11) is based on the total intensity and therefore may not reflect the local structure of the closed separating surface. In order to investigate this fine structure and the polarization of the radiation under consideration let us introduce local angular coordinates for a fixed value of t_1, which are shown in figure 1. The x-axis is directed along the tangent to the trajectory, the y-axis along the principal normal and the z-axis along the bi-normal. Then polarization direction vectors may

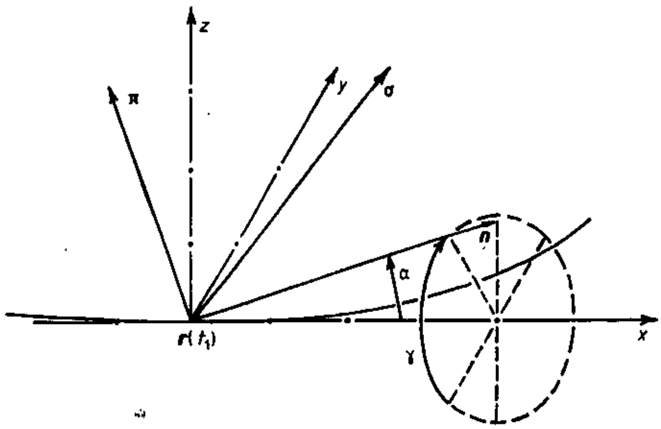

Figure 1. Local directions at the radiation point.

be introduced: $\boldsymbol{n}$ is the radiation direction, $\boldsymbol{\sigma}$ is the unit vector lying in the tangent plane and perpendicular to $\boldsymbol{n}$ and, finally, $\boldsymbol{\pi}=(\boldsymbol{n}\times\boldsymbol{\sigma})$. The angle α denotes the deviation of $\boldsymbol{n}$ from the x-axis, and γ is the azimuth in the plane perpendicular to the x-axis. The components of these vectors and of the fields are

$$\mathrm{n}=\{\cos\alpha, \sin\alpha\cos\gamma, \sin\alpha\sin\gamma, \mathrm{i}\} \tag{12}$$

$$\boldsymbol{\sigma}=\left\{-\frac{\sin\alpha\cos\gamma}{(\cos^2\alpha+\sin^2\alpha\cos^2\gamma)^{1/2}}, \frac{\cos\alpha}{(\cos^2\alpha+\sin^2\alpha\cos^2\gamma)^{1/2}}, 0\right\} \tag{13}$$

$$\boldsymbol{\pi}=\left\{\frac{-\sin\alpha\cos\alpha\sin\gamma}{(\cos^2\alpha+\sin^2\alpha\cos^2\gamma)^{1/2}}, \frac{-\sin^2\alpha\sin\alpha\cos\gamma}{(\cos^2\alpha+\sin^2\alpha\cos^2\gamma)^{1/2}}, (\cos^2\alpha+\sin^2\alpha\cos^2\gamma)^{1/2}\right\}$$

$$\boldsymbol{v}=\{v, 0, 0\} \qquad \boldsymbol{w}=\left\{0, \frac{v^2}{\alpha}, 0\right\} \tag{14}$$

$$E_n=\frac{e(1-\beta^2)}{\mathscr{R}^2(1-\beta\cos\alpha)^2} \qquad H_n=0$$

$$E_\sigma=H_\pi=\frac{e\beta^2[\beta-\cos\alpha-\beta\sin^2\alpha\sin^2\gamma]}{\mathscr{R}a(1-\beta\cos\alpha)^3(\cos^2\alpha+\sin^2\alpha\cos^2\gamma)^{1/2}}$$
$$+\frac{e\beta\sin\alpha\cos\gamma(1-\beta^2)}{\mathscr{R}^2(1-\beta\cos\alpha)^3(\cos^2\alpha+\sin^2\alpha\cos^2\gamma)^{1/2}} \tag{15}$$
$$E_\pi=-H_\sigma=\frac{e\beta^2\sin^2\alpha\sin\gamma\cos\gamma}{\mathscr{R}a(1-\beta\cos\alpha)^3(\cos^2\alpha+\sin^2\alpha\cos^2\gamma)^{1/2}}$$
$$+\frac{e\beta\sin\alpha\cos\alpha\sin\gamma(1-\beta^2)}{\mathscr{R}^2(1-\beta\cos\alpha)^3(\cos^2\alpha+\sin^2\alpha\cos^2\gamma)^{1/2}}.$$

The total intensities are

$$I_\sigma=\frac{ce^2}{2a^2}\left(\frac{\beta^4(6+\beta^2)}{6(1-\beta^2)^2}+\frac{a^2\beta^2(6-3\beta^2+\beta^4)}{6\mathscr{R}^2(1-\beta^2)}\right)$$
$$I_\pi=\frac{ce^2}{2a^2}\left(\frac{\beta^4(2-\beta^2)}{6(1-\beta^2)^2}+\frac{a^2\beta^2(2+3\beta^2-\beta^4)}{6\mathscr{R}^2(1-\beta^2)}\right). \tag{16}$$

The sum of I_σ and I_π coincides with (9).

Since

$$\mathrm{n}(\boldsymbol{E}\times\boldsymbol{H})=E_\sigma^2+E_\pi^2 \tag{17}$$

then inserting (17) into (8) we get the integrand of (8) in the form

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t_1\,\mathrm{d}\Omega}=\frac{ce^2}{4\pi a^2(1-\beta\cos\alpha)^5(\cos^2\alpha+\sin^2\alpha\cos^2\gamma)}$$
$$\times\bigg(\beta^4[(\beta-\cos\alpha-\beta\sin^2\alpha\sin^2\gamma)^2+\sin^4\alpha\sin^2\gamma\cos^2\gamma]$$
$$+\frac{2a}{\mathscr{R}}(1-\beta^2)\beta^3\sin\alpha\cos\gamma(\beta-\cos\alpha-\beta\sin^2\alpha\sin^2\gamma+\sin^2\alpha\cos\alpha\cos\gamma)$$
$$+\frac{a^2\beta^2(1-\beta^2)^2}{\mathscr{R}^2}\sin^2\alpha(\cos^2\alpha+\sin^2\alpha\cos^2\gamma)\bigg). \tag{18}$$

The term proportional to $a/\mathscr{R}$ does not contribute to the total energy flux since

$$\int_0^\pi \sin^2\alpha\,\mathrm{d}\alpha\int_0^{2\pi}\mathrm{d}\gamma\cos\gamma(\beta-\cos\alpha-\beta\sin^2\alpha\sin^2\gamma+\sin^2\alpha\cos\alpha\cos\gamma)=0.$$

This follows also from the results of Sokolov *et al* (1971). Omitting this term and equating the first and third terms in the braces in (18), we find the equation of the separatrix:

$$\mathscr{R}_s=a\frac{(1-\beta^2)\sin\alpha(\cos^2\alpha+\sin^2\alpha\cos^2\gamma)^{1/2}}{\beta[(\beta-\cos\alpha-\beta\sin^2\alpha\sin^2\gamma)^2+\sin^4\alpha\sin^2\gamma\cos^2\gamma]^{1/2}}. \tag{19}$$

The wave zone spreads at $\mathscr{R}\gg\mathscr{R}_s$; the near zone at $\mathscr{R}\ll\mathscr{R}_s$. Therefore, the exact form of the separatrix (19) is not very important, but of special interest are the following four directions: if $\alpha=0$ or $\alpha=\pi$, $\mathscr{R}_s=0$, the wave zone reaches the radiation point; if $\cos\alpha=\beta$ and $\gamma=0$ or $\gamma=\pi$, $\mathscr{R}_s=\infty$, all long-range fields vanish, the wave zone and the radiation disappear. In the former case

$$E_y=H_z=-\frac{e\beta^2}{\mathscr{R}a(1-\beta)^2}\qquad E_x=E_z=H_x=H_y=0$$
$$E_n=\frac{e(1+\beta)}{\mathscr{R}^2(1-\beta)}\qquad E_\pi=-H_\sigma=0 \tag{20}$$
$$E_\sigma=H_\pi=-\frac{e\beta^2}{\mathscr{R}a(1-\beta)^2}\qquad \frac{\mathrm{d}I}{\mathrm{d}\Omega}\bigg|_{\alpha=0}=\frac{ce^2}{4\pi a^2}\frac{\beta^4}{(1-\beta)^4}.$$

In the latter case

$$E_y=\pm\frac{e}{\mathscr{R}^2(1-\beta^2)^{3/2}}\qquad H_z=\beta E_y\qquad E_n=\frac{e}{\mathscr{R}^2(1-\beta^2)}$$
$$E_\sigma=H_\pi=\pm\frac{e\beta}{\mathscr{R}^2(1-\beta^2)^{3/2}}\qquad E_\pi=H_\sigma=0\qquad E_x=E_z=H_x=H_y=0 \tag{21}$$

The fact that in this case the border of the wave zone goes to infinity was noted first by Bagrov (1965).

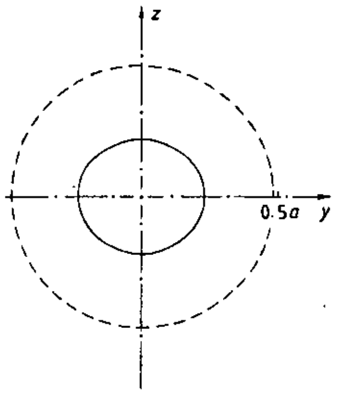

Figure 2. Section of the separatrix by the plane $x=0$.

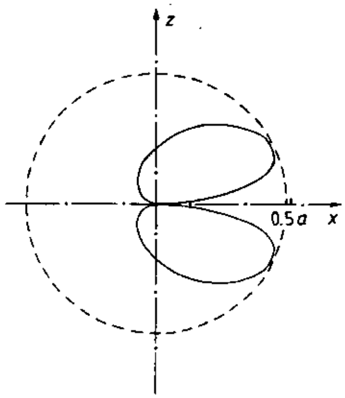

Figure 3. Section of the separatrix by the plane $y=0$.

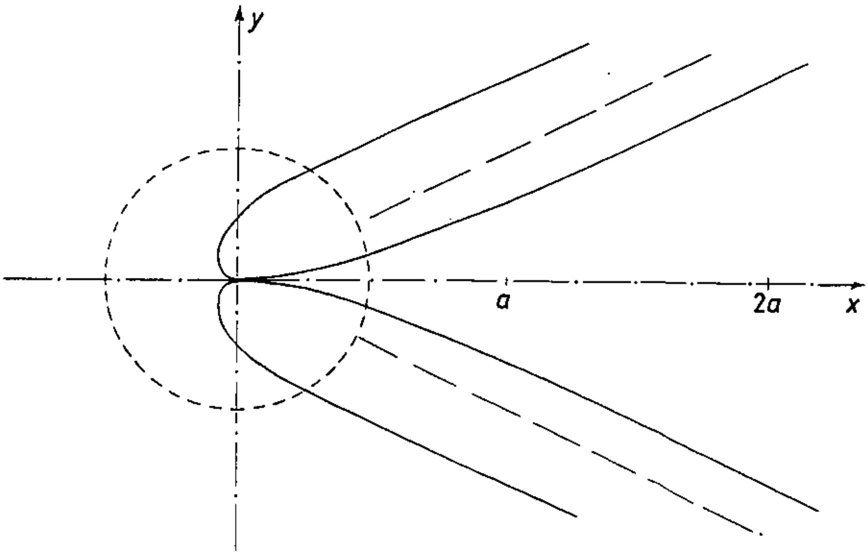

Figure 4. Section of the separatrix by the plane $z=0$.

It may be noted that integration over γ involves the integral $\int_0^{2\pi} \mathrm{d}\varphi(a^2 \sin^2 \varphi + b^2 \cos \varphi)^{-1/2} = 2\pi/|ab|$ which by Gradstein and Ryzhik (1965) is presented without the absolute value on the right-hand side, and this leads to an incorrect result.

The form of the separatrix may be illustrated by means of its sections which are shown in figures 2-4 for $\beta = 0.9$. Dotted circles show sections of the sphere (11).

Figure 2 shows the section by the plane $x = 0$ ($\alpha = \pi/2$) for which $\mathcal{R} = a(1-\beta^2)/(\beta(\beta^2 \cos^2 \gamma + \sin^2 \gamma)^{1/2})$; note the point inside. If $x < -0.6$, there is no section. If $x \approx -0.6$, there appears a section having the form of one closed curve. For $-0.6 < x < 0$, the above-mentioned curve is divided into two closed curves, one of which lies inside the other. At $x = 0$ the inner curve shrinks into a point, and the other one is shown on figure 2. If $0 < x < 0.3$, the section again consists of two curves, one of which lies inside the other, and both of them grow. For $0.3 < x < 0.44$ the outer curve acquires cavities in the z-direction from outside, and the inner one protrusions in the same direction, but from inside. At about $x \approx 0.44$ these curves touch each other and, changing their topology, break into two closed curves lying outside each other. At $0.44 < x < \infty$ these curves shift apart and become smaller.

Figure 3 shows the section of the separatrix by the plane $y = 0$ ($\gamma = \pi/2$): $\mathcal{R} = a(1-\beta^2) \sin \alpha/(\beta(1-\beta \cos \alpha))$. If $|y| > 0$, each section is a closed curve having cavities in the front and rear directions. For $|y| > 0.44$ these cavities disappear, and the sections tend to extended closed curves, becoming shorter and narrower.

Figure 4 shows the section by the plane $z = 0$ ($\gamma = 0$): $\mathcal{R} = a(1-\beta^2) \sin \alpha/(\beta|\beta - \cos \alpha|)$. If $0 < z < 0.3$, each section is a closed curve having a small cavity in the rear part and a much bigger one in the front part. For $|z| \approx 0.3$ the section shrinks to a point and disappears.

Two horns seen in figure 3 correspond to $\cos \alpha = \beta$, $\gamma = 0$ and to $\cos \alpha = \beta$ and $\gamma = \pi$. All the diagrams show that $\mathcal{R} = 0$ if $\alpha = 0$ or $\alpha = \pi$ (including the point inside figure 2).

References

Bagrov V G 1965 *Opt. Spectrosc. (USSR)* **18** 541

Gradstein I S and Ryzhik I M 1965 *Tables of Integrals, Sums, Series and Products* (New York: Academic) formula 3.642

Klepikov N P 1985a *Usp. Fiz. Nauk* **146** 317

Klepikov N P 1985b *Sov. Phys.- Usp.* **28** 506

Liénard A 1898 *L'Éclairage Électrique* **16** 5 (No 27)

Schott G A 1912 *Electromagnetic Radiation* (Cambridge: Cambridge University Press)

Schwinger J 1949 *Phys. Rev.* **75** 1912

Sokolov A A, Galtsov D V, Kolesnikova M M 1971 *Izv. Vyssh. Uchebn. Zaved. Fiz. (USSR)* **4** 14

Teitelboim C, Villaroel D, van Weert Ch G 1980 *Riv. Nuovo Cimento* **3** 1

Villaroel D and Fuenzalida V 1987 *J. Phys. A: Math. Gen.* **20** 1387

Villaroel D and Millan C 1988 *Phys. Rev.* D **38** 383